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SOLUTIONS OF A CLASS OF STATIONARY FIELDS

by

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A THESIS

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The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies and Research, for acceptance, a thesis entitled SOLUTIONS OF A CLASS OF STATIONARY FIELDS submitted by Thomas James Timothy Spanos in partial fulfillment of the requirements for the degree of Master of Science.

ABSTRACT

A class of stationary electromagnetic vacuum fields was found by Israel and Wilson[5] and independently by Perjes[6]. This work as done by Israel and Wilson[5] is given along with the general conditions for the absence of struts and magnetic poles as found by Israel and Spanos[13]. The thesis then deals with specific examples of the above fields; first multi-source axially symmetric Kerr-Newman solutions and their agreement with the general conditions for the absence of struts and magnetic poles, then a disk solution and finally spherical shell solutions and an outline of how one may find the interior Kerr-Newman solution for charge equal to mass.

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CHAPTER 1

INTRODUCTION

1.1 Content of Thesis

According to Newtonian theory, a system of bodies can be held in equilibrium by a balance between electrostatic repulsion and gravitational attraction if their charges e_i and masses m_i are related by $e_i = \varepsilon G^{\frac{1}{2}} m_i$ $(\varepsilon = \pm 1)$. In 1947 it became clear that this situation has a simple relativistic analogue when Papapetrou[1] and Majumdar[2], independently generalizing earlier work of Weyl[3], exhibited a class of static, electromagnetic vacuum fields which could be straightforwardly interpreted[4] as the exterior field of such a system.

A natural mathematical generalization of the Papapetrou-Majumdar electromagnetic vacuum fields to the stationary case (admitting sources with arbitrary angular momenta) has been given by Israel and Wilson[5] and independently by Perjes[6]. This work is covered in Chapter 2 where I follow through the discussion of stationary fields and the development of these "I.W.P. fields" as given by Israel and Wilson[5]. The I.W.P. fields are the only tool presently available for learning about the interactions between spin, gravitation and electromagnetism for arbitrarily strong fields, and

a number of detailed studies of particular solutions in the class have recently appeared[7-11]. In Chapters 4, 5 and 6 further particular solutions are dealt with; multi-source Kerr-Newman solutions, a disk solution and internal solutions.

At first sight it appears plausible to interpret the I.W.P. fields as exterior fields of stationary charged spinning sources, and this has been generally assumed. This simple interpretation is supported by i) the circumstance that the Kerr-Newman metric with $e = m$ is an I.W.P. field[5] and ii) its apparent consistency within the framework of linearized theory where it can be shown explicitly[12], that the balancing of forces extends even to the electromagnetic and gravitational spin-spin forces, at least for sources of Kerr-Newman type (gyromagnetic ratio 2).

Closer examination, however, has revealed a number of anomalies[13] which are discussed in Chapter 3. In particular the condition for equilibrium without struts or magnetic poles is found and shown to be consistent with the results of Bonnor and Ward[9]. Also in Chapter 4 this theoretical result is checked against the explicit metrics for the two and N source Kerr-Newman solutions which are found in that chapter.

In Chapter 5 I find a disk solution in which the gravitational attraction is balanced by the

electrostatic repulsion. This solution is found by following the method developed by Israel in his solution of a Kerr-Newman disk[14]. I then discuss a thin shell approximation and what seems to be an unsolvable exact thin shell to develop the formalism and types of manipulations necessary for the following chapter in which I consider the interior Kerr-Newman solutions for a thin shell and a thick shell. The final chapter also contains one possible interior extension of the I.W.P. fields[13].

A large part of this thesis is devoted to interior solutions for shells composed of material with charge equal to mass and the physical justification for looking at such solutions is straightforward. The interior Kerr-Newman metric for a thin shell with charge equal to mass should give us some insight into the form of the metric for arbitrary charge, also one may test Mach's principle[15,16] inside the shell. Inside the shell one should be able to find a global inertial frame for the shell's interior rotating with a fraction of the shell's angular velocity. An observer inside the shell would then observe the usual effects associated with "rotation" (Coriolis forces,etc.) if and only if there is a relative rotation between the local inertial axis and his frame. He can ascribe these effects either to the usual local reasons ("my reference frame is rotating relative to the

local inertial axis"), or he may consider himself at rest and the shell's rotation as generating gravitational forces on test particles which mimic Coriolis forces[17]. If extrapolation to the case $GM/c^2R \sim 1$ is permissible, we expect a limit of complete drag, with the interior inertial frames rigidly locked to the spinning shell[18,19].

1.2 Notation

In all following sections other than 2.3 relativistic units are used ($G = c = 1$). Latin indices will be used to represent a three dimensional coordinate system, in section 4.1 they run from 2 to 4 and throughout the rest of the thesis they run from 1 to 3. Greek indices will run from 1 to 4. The index 4 refers to the time coordinate and the indices 1 to 3 refer to arbitrary spatial coordinates. The summation convention is used throughout, except in equations where a summation sign appears. All terms with a bar over them are given with respect to a three dimensional space.

CHAPTER 2

I. W. P. FIELDS[5]

2.1 Stationary Fields

The metric of an arbitrary stationary field can be expressed in the form

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= f^{-1} \bar{g}_{mn} dx^m dx^n - f(\omega_m dx^m + dx^4)^2 \end{aligned} \tag{2.1}$$

in which f, $\bar{g}_{mn}$ and ω_m are independent of the time coordinate x^4 and the three-vector ω_m is arbitrary up to an additive gradient corresponding to the possibility of making time translations. The inverse of $g_{\mu\nu}$ is given by

$$\begin{aligned} g^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial x^\nu} = f\bar{g}^{mn} \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^n} - 2f\omega^m \frac{\partial}{\partial x^m} \frac{\partial}{\partial x^4} - \\ - (f^{-1} - f\omega^2) \frac{\partial^2}{(\partial x^4)^2} \end{aligned} \tag{2.2}$$

where $\bar{g}^{mn}$ is the 3×3 symmetric matrix inverse to $\bar{g}_{mn}$, $\omega^m = \bar{g}^{mn}\omega_n$ and $\omega^2 = \bar{g}^{mn}\omega_m\omega_n$. The determinants of $g_{\mu\nu}$ and $\bar{g}_{mn}$ are related by

$$(-g)^{\frac{1}{2}} = f^{-1}\bar{g}^{\frac{1}{2}} \quad . \tag{2.3}$$

From the 3-vector ω_m we can derive an invariant "torsion vector"

$$f^{-2}\underset{\sim}{\tau} = \text{curl}\ \underset{\sim}{\omega} \tag{2.4}$$

in terms of a three dimensional vector calculus employing $\bar{g}_{mn}dx^m dx^n$ as base metric.

Now consider a stationary electromagnetic field

$$F_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$$

in the space time (2.1). Therefore we may write

$$F_{4n} = \partial_n A_4 - \partial_4 A_n$$

and the condition of time independence, $\partial_4 A_\mu = 0$, yields for the "electric" components

$$F_{4n} = \partial_n A_4 \quad . \tag{2.5}$$

The source-free Maxwell equations

$$\partial_n[(-g)^{\frac{1}{2}}\ {}^{(4)}F^{\mu n}] = 0 \tag{2.6}$$

for $\mu = m$ give

$$\partial_n[(-g)^{\frac{1}{2}}\ {}^{(4)}F^{mn}] = 0$$

and thus the "magnetic" components

$${}^{(4)}F^{mn} = f\,\bar{g}^{-\frac{1}{2}}\varepsilon^{mnp}\partial_p\phi \tag{2.7}$$

in terms of a magnetic scalar potential ϕ. All remaining components are then conveniently expressed in terms of these six; for example

$$^{(4)}F^{n4} = \omega_m {}^{(4)}F^{mn} + F_{4m}\bar{g}^{mn} \tag{2.8}$$

Equation (2.6) with $\mu = 4$ yields

$$\partial_n[(-g)^{\frac{1}{2}} \, {}^{(4)}F^{4n}] = 0 \ ,$$

on substituting (2.8) one obtains

$$\partial_n[(-g)^{\frac{1}{2}}(\omega_m {}^{(4)}F^{mn} + {}^{(4)}F_{4m}\bar{g}^{mn})] = 0$$

and substituting (2.5) and (2.7) this becomes

$$\partial_n[(-g)^{\frac{1}{2}}(\omega_m f\bar{g}^{-\frac{1}{2}}\varepsilon^{mnp}\partial_p\Phi + \bar{g}^{mn}\partial_m A_4)] = 0 \ .$$

Now using equation (2.3), substituting the torsion vector into the first term and recognizing that the second term is a divergence we obtain

$$\operatorname{div}(f^{-1}\nabla A_4) = -f^{-2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nabla}\Phi \ . \tag{2.9}$$

Next writing F_{mn} in terms of (2.5) and (2.7) and expressing the cyclic identity

$$\varepsilon^{mnp}\partial_p F_{mn} = 0 \ ,$$

we obtain

$$\operatorname{div}(f^{-1}\nabla\Phi) = f^{-2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nabla}A_4 \ . \tag{2.10}$$

If we now introduce the complex scalar potential

$$\Psi = A_4 + i\Phi \ , \tag{2.11}$$

then (2.9) and (2.10) combine to give

$$\operatorname{div}(f^{-1}\nabla\Psi) = if^{-2}\underset{\sim}{\tau}\cdot\underset{\sim}{\nabla}\Psi \quad . \tag{2.12}$$

We have thus reduced the entire set of Maxwell's equations to the single complex equation (2.12).

2.2 Gravitational Field Equations

The Ricci tensor for the general stationary metric (2.1) is conveniently expressed in terms of a complex 3-vector $\underset{\sim}{G}$, defined by

$$2f\underset{\sim}{G} = \underset{\sim}{\nabla}f + i\underset{\sim}{\tau} \quad . \tag{2.13}$$

Then[20,21]

$$-f^{-2}R_{44} = \operatorname{div}\underset{\sim}{G} + (\underset{\sim}{G}^* - \underset{\sim}{G})\cdot\underset{\sim}{G} \ , \tag{2.14a}$$

$$-if^{-2}\,{}^{(4)}R^m_4 = \bar{g}^{-\frac{1}{2}}\varepsilon^{mpq}(\partial_q G_p + G_p G^*_q) \tag{2.15a}$$

$$f^{-2}(\bar{g}_{pm}\bar{g}_{qn}\,{}^{(4)}R^{mn} - \bar{g}_{pq}R_{44}) = R_{pq}(\bar{g}) + G_pG^*_q + G^*_pG_q \ . \tag{2.16a}$$

Here $R_{pq}(\bar{g})$ denotes the Ricci tensor formed from the 3-metric $\bar{g}_{mn}dx^m dx^n$.

For the electromagnetic energy tensor

$$-4\pi T_{\mu\nu} = g^{\alpha\beta}F_{\mu\alpha}F_{\nu\beta} - \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}$$

one derives from the formulas of section 2.1

$$\frac{1}{2}F_{\mu\nu}F^{\mu\nu} = (\nabla\phi)^2 - (\nabla A_4)^2 \ ,$$

$$8\pi f^{-1}T_{44} = (\nabla\phi)^2 + (\nabla A_4)^2 \ , \tag{2.14b}$$

$$4\pi f^{-1}\ {}^{(4)}T^m_4 = \bar{g}^{-\frac{1}{2}}\varepsilon^{mpq}(\partial_p\phi)(\partial_4 A_4) \quad , \tag{2.15b}$$

$$-4\pi f^{-1}\ {}^{(4)}T^{mn} = (\partial^m\phi)(\partial^n\phi) + (\partial^m A_4)(\partial^n A_4) - $$
$$- \frac{1}{2}\,\bar{g}^{mn}[(\nabla\phi)^2 + (\nabla A_4)^2] \ , \tag{2.16b}$$

with $\partial^m = \gamma^{mn}\partial_n$.

We can now impose the Einstein field equations

$$R_{\mu\nu} = -8\pi\ T_{\mu\nu}\ .$$

Thus from (2.15a) and (2.15b) we have

$$\bar{g}^{-\frac{1}{2}}\varepsilon^{mpq}(\partial_q G_p + G_p G_q^*) = i\,2\,f^{-1}\bar{g}^{-\frac{1}{2}}\varepsilon^{mpq}(\partial_p\phi)(\partial_q A_4)$$

and upon substituting (2.13) and using the anti-symmetry properties of ε^{mpq} we obtain

$$\text{curl}\ \underset{\sim}{\tau} = -4\underset{\sim}{\nabla}\Phi \times \underset{\sim}{\nabla}A_4$$
$$= i\ \text{curl}(\Psi\nabla\Psi^* - \Psi^*\nabla\Psi) \ ,$$

so that the equation

$$\underset{\sim}{\tau} + i(\Psi^*\underset{\sim}{\nabla}\Psi - \Psi\underset{\sim}{\nabla}\Psi^*) = \underset{\sim}{\nabla}\psi \tag{2.17}$$

defines a real scalar ψ up to an additive constant.

We next define a complex function

$$\varepsilon = f - \Psi\Psi^* + i\psi \quad . \tag{2.18}$$

By virtue of (2.13) and (2.17) we obtain,

$$f\underset{\sim}{G} = \frac{1}{2}\underset{\sim}{\nabla}\varepsilon + \Psi^*\nabla\Psi \quad . \tag{2.19}$$

Substituting (2.19) into the field equation (2.14a), (2.14b) and employing (2.12) leads to

$$f\nabla^2\varepsilon = \underset{\sim}{\nabla}\varepsilon\cdot(\underset{\sim}{\nabla}\varepsilon + 2\Psi^*\nabla\Psi) \ , \tag{2.20}$$

while (2.12) itself can be written

$$f\nabla^2\Psi = \underset{\sim}{\nabla}\Psi\cdot(\underset{\sim}{\nabla}\varepsilon + 2\Psi^*\nabla\psi) \ , \tag{2.21}$$

and we note from (2.18) that

$$f = \frac{1}{2}(\varepsilon + \varepsilon^*) + \Psi\Psi^* \ . \tag{2.22}$$

Finally the field equations (2.16a), (2.16b) reduce to

$$-f^2R_{mn}(\bar{g}) = \frac{1}{2}\,\varepsilon_{,(m}\varepsilon^*_{,n)} + \Psi\varepsilon_{,(m}\Psi^*_{,n)} + \Psi^*\varepsilon_{,(m}\Psi_{,n)}$$

$$- (\varepsilon + \varepsilon^*)\Psi_{,(m}\Psi^*_{,n)} \quad . \tag{2.23}$$

The complete system of electromagnetic and gravitational field equations for an arbitrary electromagnetic vacuum field are summed up in (2.20), (2.21) and (2.23).

2.3 I.W.P. Fields

We now examine the solutions of the systems (2.20), (2.21) and (2.23) for which the background metric $\bar{g}_{mn}dx^m dx^n$ is flat. f and $\underset{\sim}{\omega}$ may be obtained from the equations[13]

$$f = (UU^*)^{-1}$$
$$i \operatorname{curl} \underset{\sim}{\omega} = U^*\nabla U - U\nabla U^* \tag{2.24}$$

where U is any solution of the equation

$$U\nabla^2 U^* = U^*\nabla^2 U$$

and the space-time metric is given by

$$ds^2 = f^{-1}(dx^2+dy^2+dz^2) - f(dt+\omega_a dx^a)^2 \ . \tag{2.25}$$

The electric field may then be described by the self-dual electromagnetic field tensor

$$\mathcal{F}_{\alpha\beta} = F_{\alpha\beta} + i\,{}^{\dagger}F_{\alpha\beta}$$
$$= (i\varepsilon_{abc}e_\alpha^{(a)}e_\beta^{(b)} + e_\alpha^{(c)}v_\beta - e_\beta^{(c)}v_\alpha)U^{-2}\partial_c U \tag{2.26}$$

where

$$e_\alpha^{(a)} = f^{-\frac{1}{2}}\delta_\alpha^a \quad ;$$
$$v^\alpha = f^{-\frac{1}{2}}\delta_4^\alpha$$

and

$${}^{\dagger}F_{\alpha\beta} = \frac{1}{2}\,(-g)^{\frac{1}{2}}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu} \qquad .$$

This is equivalent to the description given by (2.5) and (2.7) where

$$\psi = (1 - \frac{1}{U})\ e^{i\alpha} \ .$$

Any constant phase transformation

$$U \to U e^{i\alpha} \qquad (\alpha \text{ const})$$

leaves the metric invariant and induces a constant duality rotation of the electromagnetic field,

$$\mathcal{F}_{\alpha\beta} \to \mathcal{F}_{\alpha\beta} e^{-i\alpha} .$$

The I.W.P. result can now be stated as follows: the Einstein-Maxwell field described by (2.25) and (2.26) is an electromagnetic vacuum field (i.e. locally free of charge and material sources) if $\nabla^2 U = 0$.

2.4 The Linearized Theory[23-26]

Assume the gravitational field to be weak in the sense $V/c^2 << 1$ (V = Newtonian potential). We may introduce coordinates which are almost rectilinear, therefore $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, $\eta_{\mu\nu} \equiv \mathrm{diag}(1,1,1,-1)$ and $|h_{\mu\nu}| << 1$ (i.e. the metric tensor differs only slightly from the flat space metric tensor). Only terms up to first order in $h_{\mu\nu}$ are kept in the equations and raising and lowering of indices is done with $\eta_{\mu\nu}$. In the case of linearized stationary fields we have

$$h_{\mu\nu} = \frac{4G}{c^4} \int \frac{T_{\mu\nu}(\underset{\sim}{r}') - \frac{1}{2}\eta_{\mu\nu} T(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} d^3r' \quad .$$

Assume that the motion of the sources is non-relativistic, i.e. $|u^i/c| << 1$, then the energy tensor is given by

$$T^{\mu\nu} = \mu u^\mu u^\nu$$

where

$$u^4 = c + 0(h) ,$$

$T = -\mu c^2$, $T^{i4} = \mu c u^i$ and T^{ij}/T is negligible ($T = T^\mu_\mu$). (We must set the stresses equal to zero for consistency in the linear approximation).

Write

$$V(\underset{\sim}{r}) = -G \int \frac{\mu(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} d^3r'$$

$$A_i(\underset{\sim}{r}) = 4G \int \frac{\mu(\underset{\sim}{r}')u_i(\underset{\sim}{r}')}{|\underset{\sim}{r} - \underset{\sim}{r}'|} d^3r' ,$$

therefore $h_{11} = h_{22} = h_{33} = h_{44} = -2V/c^2$, $h_{i4} = h_{4i} = -A_i/c^3$ and $h_{12} = \ldots = 0$. Hence

$$ds^2 = (1 - \frac{2V}{c^2})(dx^2 + dy^2 + dz^2) - (1 + \frac{2V}{c^2})(cdt + \frac{1}{c^3} A_i dx^i)^2 .$$

Thus in the linearized theory we have

$$f^{-1} \to (1 - \frac{2V}{c^2}) , \quad f \to (1 + \frac{2V}{c^2}) \quad \text{and} \quad \omega_i \to \frac{A_i}{c^3} .$$

For the case of a single axially symmetric Kerr-Newman source we have that U in linearized form becomes $1 + \frac{m}{r}$, then $f^{-1} = 1 + \frac{2m}{r}$, $f = 1 - \frac{2m}{r}$ and $\omega_\phi = \frac{2ma}{r} \sin^2\theta$ in spherical coordinates. This is consistent with the previous result since $V = - \frac{GM}{r}$ where M is the total mass,

$m = \frac{GM}{c^2}$ and $a \equiv \frac{GL}{mc^3}$ where L is the angular momentum. By this simple example one may get a feeling for how the I.W.P. formalism is linearized.

CHAPTER 3

EQUILIBRIUM IN AN I.W.P. FIELD

3.1 Difficulties encountered with the Solution

A very interesting result arose when Bonnor and Ward[9] calculated the explicit form of the I.W.P. metric generated by

$$U = 1 + \sum_{A=1}^{2} \frac{q_A}{r_A} + \frac{iq_A a_A (z + (-1)^A b)}{r_A^3} \tag{3.1}$$

which represents two sources ("Perjeons") on the z-axis with spins and magnetic moments parallel to the z-axis. Here $r_A^2 = \rho^2 + (z + (-1)^A b)^2$, $\rho^2 = x^2 + y^2$, q_A = mass = charge and a_A is the angular momentum per unit mass of particle A. They solved for ω in (2.24) to obtain[33]

$$\omega_\phi = \left\{ \frac{q_1 a_1}{r_1^3}\left(2 + \frac{q_1}{r_1}\right) + \frac{q_2 a_2}{r_2^3}\left(2 + \frac{q_2}{r_2}\right) + \frac{q_1 q_2 a_2}{\rho^2}\left[\frac{r_1}{2b^2} + \frac{(z+b)(\rho^2+z^2-b^2)}{br_1 r_2}\right] \right.$$

$$\left. + \frac{q_1 q_2 a_1}{\rho^2}\left[\frac{r_2}{2b^2 r_1} - \frac{(z-a)(\rho^2+z^2-b^2)}{br_2 r_1^3}\right] + \frac{K}{\rho^2} \right\} \tag{3.2}$$

where K is an arbitrary constant and the angle ϕ lies in the x-y plane. Whatever the choice of K, this solution for ω causes the metric to be singular somewhere on the rotation axis unless

$$a_1 + a_2 = 0 \ , \tag{3.3}$$

a restriction Bonnor and Ward were unable to interpret physically. When this condition is not satisfied the metric may be made non-singular on the axis either for $|z| > b$ or $|z| < b$ by a suitable choice of K but not for both. With the choice

$$K = -(a_1 + a_2)/2b^2$$

they make the metric non-singular for $|z| > b$ but singular for $|z| < b$ unless (3.3) is satisfied. Substituting this value of K into (3.2) we obtain

$$\omega_\phi = \left\{ \frac{L_1}{r_1^3}(2 + \frac{m_1}{r_1}) + \frac{L_2}{r_2^3}(2 + \frac{m_2}{r_2}) + \frac{m_1 L_2}{\rho^2}[-\frac{1}{2b^2} + \frac{r_1}{2b^2 r_2} + \frac{(z-b)(\rho^2+z^2-b^2)}{b r_1 r_2^3}] \right.$$

$$\left. + \frac{m_2 L_1}{\rho^2}[-\frac{1}{2b^2} + \frac{r_2}{2b^2 r_1} - \frac{(z-b)(\rho^2+z^2-b^2)}{b r_2 r_1^3}] \right\} \ . \tag{3.4}$$

Thus (3.1), (3.4), (2.24), (2.25) and (2.26) form the complete I.W.P. solution in this case.

Hartle and Hawking[7] have studied solutions involving complex parameters q_A and also found that special restrictions are needed to avoid line singularities.

Robert Wald[27] has found a further difficulty which arises when one considers a stationary, non-spinning test particle with $e = m$ in an I.W.P. field:

Unless the field is static ($U^* = U$), equilibrium is generally impossible.

3.2 How these Difficulties are overcome.[13]

The Wald paradox may be overcome by introducing magnetic charge μ to the test particle. The equations of motion

$$m\left(\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\,\frac{dx^\alpha}{ds}\,\frac{dx^\beta}{ds}\right) = (eF^\lambda{}_\mu + \mu\,{}^\dagger F^\lambda{}_\mu)\,\frac{dx^\mu}{ds}$$

reduce for a stationary particle to

$$m\nabla(f^{\frac{1}{2}}) = \mathrm{Re}\{(e - i\mu)\nabla(U^{-1})\}$$

and are satisfied if $e - i\mu = m(U/U^*)^{\frac{1}{2}}$. (3.5)

This example illustrates that the possibility of magnetic poles is a factor to be reckoned with in the study of stationary interactions of spinning charges. When a 2-surface s encloses sources we have $F^{\lambda\mu}{}_{,\mu} = 0$ and $F_{\lambda\mu,\nu} + F_{\nu\lambda,\mu} + F_{\mu\nu,\lambda} = 0$ since we have an electromagnetic vacuum field. Thus

$$\mathcal{F}^{\lambda\mu}{}_{,\mu} = F^{\lambda\mu}{}_{,\mu} + i\,{}^\dagger F^{\lambda\mu}{}_{,\mu} = 0$$

and for $\lambda = 4$ we have

$$\mathcal{F}^{4a}{}_{,a} = 0\ ,$$

therefore

$$\int_V (-g)^{\frac{1}{2}} \mathcal{F}^{4a}{}_{,a}\, dV = 0$$

$$= \int_{s_1+s_2} (-g)^{\frac{1}{2}} \mathcal{F}^{4a} n_a ds$$

which may be rewritten

$$\int_{s_1} (-g)^{\frac{1}{2}} \mathcal{F}^{4a} n_a' ds = \int_{s_2} (-g)^{\frac{1}{2}} \mathcal{F}^{4a} n_a ds \quad .$$

Here s_1 encloses the source, s_2 encloses s_1 with a volume V between the two surfaces, n_a' is the outward normal on s_1 and n_a is used to represent both the outward normal on s_2 and the inward normal on s_1. The electromagnetic vacuum equations thus require only that the complex flux integral

$$\Phi = \Phi_{el} + i\Phi_{mag} = \int_s (-g)^{\frac{1}{2}} \mathcal{F}^{4a} n_a ds$$

$$= - \int_s (\underset{\sim}{\nabla} U^* - i \text{ curl } U^{-1}\underset{\sim}{\omega})\cdot\underset{\sim}{n} ds \qquad (3.6)$$

must be invariant under continuous deformations of the closed 2-surface s in the exterior space, not that ϕ_{mag} must vanish when s encloses sources.

For a single source ϕ_{mag} can be trivially reduced to zero by a constant duality rotation; if there is more than one source, $\phi_{mag} = 0$ imposes non-trivial constraints which (in the case where $U^{-1}\underset{\sim}{\omega}$ is regular on s) yield

$$
\begin{aligned}
\phi_{el} &= -\int_S \nabla U^* \cdot \underset{\sim}{n} ds \\
&= -\int_S \frac{\partial}{\partial n} \left(1 + \frac{q_1^*}{r} H_o + \frac{q_2^*}{r^2} H_1 + \ldots\right) ds \\
&= 4\pi\, q_1^*
\end{aligned}
$$

and essentially require that the monopole moments q_A in any spherical harmonic expansion $H_o, H_1, \ldots.$ of U should be real.

These considerations on magnetic flux of course have no bearing on the question whether the exterior geometry is non-singular. For this, a necessary condition appears to be the regularity of the gravomagnetic potential $\underset{\sim}{\omega}$. (Hartle and Hawking[7] have shown that singularities of $\underset{\sim}{\omega}$ generally entail geometrical singularities of the charged N.U.T. type.) Assuming the regularity of $\underset{\sim}{\omega}$ in the exterior region, applying Stokes theorem to an exterior 2-surface, and recalling (2.24) we obtain the vanishing of the "gravomagnetic flux"

$$
\int_S (U^* \nabla U - U \nabla U^*) \cdot \underset{\sim}{n} ds = 0 \qquad (3.7)
$$

as a necessary condition for the regularity of the exterior geometry. If U itself is regular and non-vanishing in the exterior region, then the conditions (3.7) (for an arbitrary exterior closed surface) are also sufficient for regularity.

If s encloses a single body B (complex potential U_B, $|U_B| \to 1$ at infinity) in an external field with potential U_E ($U_E \to 0$ at infinity), (3.7) yields

$$\int_{(\text{infinity})} (U_B^* \nabla U_B - U_B \nabla U_B^*) \cdot \underset{\sim}{n} ds + \int_B (U_E^* \nabla^2 U_B - U_E \nabla^2 U_B^*) dV = 0 \tag{3.8}$$

since $\nabla^2 U_B = 0$ outside s and $\nabla^2 U_E = 0$ inside s. The surface integral is just the imaginary part of the monopole component q of U_B. For an isolated body, (3.8) requires this to vanish. In the presence of external bodies, (3.8) determines the imaginary part of q needed to ensure equilibrium without geometrical singularities.

We conclude that any configuration of I.W.P. sources having arbitrary masses and angular momenta can be held in equilibrium without external struts. Normally, however, this will require each body to carry magnetic charge. If the exterior geometry is regular, (3.6) shows that the electric and magnetic charges e, μ on a source B are given by

$$e - i\mu = -\frac{1}{4\pi} \int \nabla U_B \cdot \underset{\sim}{n} ds = q \quad . \tag{3.9}$$

In the case of the Bonnor-Ward solution, equation (3.8) reduces (for real q_A) precisely to (3.3), which is therefore the condition for achieving equilibrium without the aid of struts or magnetic poles.

CHAPTER 4

KERR-NEWMAN SOURCES

4.1 Equilibrium

For a Kerr-Newman source

$$U_B = 1 + q[(\underset{\sim}{r}-\underset{\sim}{r}_B)^2]^{-\frac{1}{2}} \qquad (q,\ \underset{\sim}{r}_B \text{ complex}) \tag{4.1}$$

in an external field, (3.8) yields a rigorous analogue of (3.5):

$$q^* - q = q U_E^*(\underset{\sim}{r}_B) - q^* U_E(\underset{\sim}{r}_B^*) \quad . \tag{4.2}$$

Equilibrium without magnetic poles is thus possible for configurations in which

$$U_E^*(\underset{\sim}{r}_B) = U_E(\underset{\sim}{r}_B^*) \quad . \tag{4.3}$$

For a system of two Kerr-Newman sources A, B this is equivalent to

$$\frac{q_A}{[(\underset{\sim}{r}_B - \underset{\sim}{r}_A^*)^2]^{\frac{1}{2}}} = \frac{q_A}{[(\underset{\sim}{r}_B^* - \underset{\sim}{r}_A)^2]^{\frac{1}{2}}} \tag{4.4}$$

where $\underset{\sim}{r}_\xi = \underset{\sim}{b}_\xi + iq_\xi^A$ $(\xi = A,B)$, $-\underset{\sim}{b}_\xi$ is the position vector and $-a_\xi$ is the angular momentum per unit mass of particle ξ. Since $(\underset{\sim}{r}_B - \underset{\sim}{r}_A^*)^2$ and $(\underset{\sim}{r}_B^* - \underset{\sim}{r}_A)^2$ are complex the equation (4.4) tells us that they are equal, i.e.

$$\mathrm{Im}(\underset{\sim}{r}_B - \underset{\sim}{r}_A^*)^2 = 0 \quad .$$

Therefore the condition for two Kerr-Newman sources to be held in equilibrium without the aid of struts or poles is[28]

$$(\underset{\sim}{b}_B - \underset{\sim}{b}_A)\cdot(\underset{\sim}{a}_B + \underset{\sim}{a}_A) = 0 \; . \tag{4.5}$$

For a system of N Kerr-Newman sources (4.3) becomes the N equations,

$$\sum_{\substack{A=1\\A\neq B}}^{N} \frac{q_A}{[(\underset{\sim}{r}_B-\underset{\sim}{r}_A^*)^2]^{\frac{1}{2}}} = \sum_{\substack{A=1\\A\neq B}}^{N} \frac{q_A}{[(\underset{\sim}{r}_B^*-\underset{\sim}{r}_A)^2]^{\frac{1}{2}}} \qquad (B=1,\ldots,N) \tag{4.6}$$

and thus

$$\mathrm{Im} \sum_{\substack{A=1\\A\neq B}}^{N} \frac{q_A}{[(\underset{\sim}{r}_A^*-\underset{\sim}{r}_B)^2]^{\frac{1}{2}}} = 0 \quad . \tag{4.7}$$

In the case where the particles all lie on a common axis with parallel angular momenta oriented along the common axis (4.7) becomes

$$\sum_{A=1}^{N} q_A \;\epsilon(A-B)\; A_{AB} = 0 \tag{4.8}$$

where

$$A_{AB} = \frac{a_A + a_B}{(b_A-b_B)^2 + (a_A+a_B)^2} \; , \quad a_\xi = |a_\xi| \; , \quad b_\xi = |b_\xi|$$

and

$$\epsilon(A-B) = \begin{cases} 1 & \text{if} \quad A-B>0 \\ 0 & \text{if} \quad A-B=0 \\ -1 & \text{if} \quad A-B<0 \end{cases} .$$

4.2 A Solution for Two Sources[11]

When we set

$$U = 1+\frac{q}{R} \quad \text{where} \quad R^2 = (\underset{\sim}{r}-\underset{\sim}{r}_B)^2 = x^2+y^2+(z-ia)^2$$

equations (2.24), (2.25) and (2.26) yield the charged Kerr-Newman solution with $e^2 = m^2$ [5] where e and m are the charge and mass respectively.

If we consider an axisymmetric generalization representing two Kerr-Newman sources with spins aligned along the connecting line, we then set

$$U = 1 + \sum_{A=1}^{2} \frac{q_A}{R_A} \tag{4.9}$$

where $R_A^2 = \rho^2 + z_A^2$; $\rho^2 = x^2 + y^2$ and $z_A^2 = (z + (-1)^A b + ia_A)^2$. Thus in the Euclidean map we have source A located on the z-axis at the position $(-1)^{A+1}b$, with angular momentum per unit mass $-a_A$, $m_A^2 = e_A^2$ where m_A and e_A are the mass and charge of particle A respectively and the charges have the same sign.

A special case of this solution has been found by Parker, Ruffini and Wilkins in which $q_1 = q_2$ and $a_1 = -a_2$ [8].

Due to the axial symmetry we have

$$\omega_n dx^n = \omega_\phi d\phi \tag{4.10}$$

where ω_ϕ is a covariant component and the angle ϕ is measured in the x-y plane. Applying Stokes theorem on an annulus of constant z concentric with the axis and of infinite outer radius we obtain:

$$\begin{aligned} \omega_\phi &= \frac{1}{2\pi} \int \text{curl } \underset{\sim}{\omega} . \underset{\sim}{n} \; ds \\ &= \text{Im} \int U \frac{\partial U^*}{\partial z} d(\rho^2) \qquad \text{from (2.24)} \end{aligned} \tag{4.11}$$

where by an appropriate choice of the additive constant we may fix the additive gradient such that ω tends to zero as ρ approaches infinity. This solution must also be consistent with

$$\frac{\partial \omega_\phi}{\partial z} = - 2\rho \text{ Im } U \frac{\partial U^*}{\partial \rho} \tag{4.12}$$

which comes directly from (2.24) also.

Upon integration one obtains

$$\omega_\phi = 2 \text{ Im} \sum_{A=1}^{2} \left(\frac{q_A \overset{*}{z}_A}{\overset{*}{R}_A} + \sum_{B=1}^{2} \frac{q_A q_B \overset{*}{z}_B R_A}{\overset{*}{z}{}_A^2 - \overset{*}{z}{}_B^2 \overset{*}{R}_B} \right) + C(z) \; . \tag{4.13}$$

Substituting (4.4), (4.10) and (4.13) into (2.25) and (2.26) we obtain the solution of the Einstein-Maxwell equations for two charged Kerr-Newman sources with $e_A^2 = m_A^2$ and sign $(e_1) = \text{sign}(e_2)$.

Write the second term of (4.13) as

$$Q = 2 \text{ Im} \sum_{A=1}^{2} \sum_{B=1}^{2} q_A q_B G_{AB} \tag{4.14}$$

where

$$G_{AB} \equiv \frac{\overset{*}{z}_A R_B}{(\overset{*}{z}{}_A^2 - a_B^2) R_A} \quad . \tag{4.15}$$

Then

$$Q = \sum_{A=1}^{2} \sum_{B=1}^{2} q_A q_B (G_{AB} - \overset{*}{G}_{BA})$$

$$= \sum_{A=1}^{2} \sum_{B=1}^{2} q_A q_B \frac{\rho^2 + \overset{*}{z}_A z_B}{(\overset{*}{z}_A - z_B) R_A \overset{*}{R}_B} \tag{4.16}$$

and using this form of the second term in equation (4.13) we compute $\partial\omega_\phi/\partial z$ and compare the result with equation (4.12), which gives that the constant of integration is independent of z[29].

Regularity on the z-axis requires our solution to satisfy

$$\omega_\phi(0,z) = K \tag{4.17}$$

where K is a constant. From (4.13) we obtain

$$\omega_\phi(0,z) = \sum_{A=1}^{2} \sum_{\substack{B=1 \\ B \neq A}}^{2} \frac{q_A q_B (a_A + a_B)}{(b_A - b_B)^2 + (a_A + a_B)^2} \frac{1}{\varepsilon_A \varepsilon_B} + \sum_{A=1}^{2} \frac{q_A^2}{2a_A} + C \tag{4.18}$$

where

$$\varepsilon_A = \begin{cases} +1 & \text{for } z - b_A > 0 \\ -1 & \text{for } z - b_A < 0 \end{cases}$$

and setting the first term in equation (4.18) equal to zero is equivalent to the general condition for the absence of struts and poles. Then

$$K = \sum_{A=1}^{2} \frac{q_A^2}{2a_A} + C$$

and if we set

$$C = -\sum_{A=1}^{2} \frac{q_A^2}{2a_A}$$

we have ω_ϕ going to zero on the z axis and as ρ approaches infinity.

From (4.13) we have that ω_ϕ is an analytic function of ρ^2,

$$\left.\frac{\partial \omega_\phi}{\partial \rho}\right|_{\rho=0} = 0 \ , \tag{4.19}$$

which, with the above choice of C, combines with the previous result to yield a more restrictive condition for regularity, that ω_ϕ approach zero faster than ρ.

If one sets $-a_1 = a_2$, $q_1 = q_2$ and considers two redundant sets of oblate spheroidal coordinates centered at each of the sources then apart from a sign[34] ω_ϕ may be transformed into the form obtained by Parker, Ruffini and Wilkins[8].

As checks on the metric it may be easily verified that ω_ϕ vanishes in the limit as a_1 and a_2 approach zero and if we fix source one at the origin

and let source two go off to infinity one finds

$$\omega_\phi \longrightarrow \frac{(q_1^2 - 2q_1r_1)\ a_1 \sin^2\theta_1}{(r_1 - q_1)^2 + a_1^2 \cos^2\theta_1} \tag{4.20}$$

which is the solution for a single Kerr-Newman source with $e_1^2 = m_1^2$ and angular momentum per unit mass directed along the z axis[5].

4.3 A Solution for N Kerr-Newman Sources

If we now consider the generalization of the solution in 4.2 to that of N Kerr-Newman sources with spins aligned along the connecting line, we then set

$$U = 1 + \sum_{A=1}^{N} \frac{q_A}{R_A} \tag{4.21}$$

where $R_A^2 = \rho^2 + z_A^2$; $\rho^2 = x^2 + y^2$ and $z_A^2 = (z + b_A + ia_A)^2$. Thus in the Euclidean map we now have source A located on the z-axis at the position $-b_A$, with angular momentum per unit mass $-a_A$, $m_A^2 = e_A^2$ and the charges have the same sign.

Substituting (4.19) into (4.11) and integrating one obtains

$$\omega_\phi = 2\ \mathrm{Im} \sum_{A=1}^{N} \frac{q_A \overset{*}{z}_A}{\overset{*}{R}_A} + \sum_{A=1}^{N} \sum_{B=1}^{N} q_A q_B \frac{\rho^2 + \overset{*}{z}_A z_B}{(\overset{*}{z}_A - z_B) R_A \overset{*}{R}_B} + C \quad . \tag{4.22}$$

(See Appendix A).

Substituting (4.21), (4.10) and (4.22) into (2.25) and (2.26) we obtain the solution of the Einstein-Maxwell

equations for N charged Kerr-Newman sources with $e_A^2 = m_A^2$ and sign (e_A) = sign (e_B) $\forall$ $A,B \in \{1,\ldots,N\}$.

From equation (4.22) we obtain

$$\omega(0,z) = \sum_{A=1}^{N} \sum_{\substack{B=1 \\ B\neq A}}^{N} \frac{q_A q_B (a_A + a_B)}{(b_A - b_B)^2 + (a_A + a_B)^2} \frac{1}{\varepsilon_A \varepsilon_B} + \sum_{A=1}^{N} \frac{q_A^2}{2a_A} + C \tag{4.23}$$

where ε_A are as defined previously. It seems obvious that we should expect the following generalization of the two source solution, that

$$K = \sum_{A=1}^{N} \frac{q_A^2}{2a_A} + C$$

and that the first term should be equivalent to the general condition for the absence of struts and poles in this N source alignment. This in fact turns out to be exactly what happens. (See Appendix A).

CHAPTER 5

SHELL SOLUTIONS

5.1 Boundary Conditions

If we have a shell, $\sum$, we may define the co-ordinates of the four-dimensional space to be x^α and on $\sum$ to be θ^a. Now defining $\underset{\sim}{e}_{(a)}$ and g_{ab} by $g_{ab} = \underset{\sim}{e}_{(a)} \cdot \underset{\sim}{e}_{(b)} = g_{\alpha\beta} e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta}$ and $e_{(a)}{}^{\alpha} = \partial x^\alpha / \partial \theta^a$, we have $d\underset{\sim}{s} = \underset{\sim}{e}_{(a)} d\theta^a$ and thus

$$
\begin{aligned}
ds^2\Big|_\Sigma &= g_{\alpha\beta} dx^\alpha dx^\beta \\
&= g_{\alpha\beta} e_{(a)}{}^{\alpha} e_{(b)}{}^{\beta} d\theta^a d\theta^b \\
&= g_{ab} d\theta^a d\theta^b
\end{aligned}
$$

where $g^{ac} g_{bc} = \delta^a{}_b$; $e^{(a)\alpha} = g^{ab} e_{(b)}{}^{\alpha}$, $e^{(a)\alpha} e_{(b)\alpha} = \delta^a{}_b$ and $g_{\alpha\beta} e^{(a)\alpha} e^{(b)\beta} = g^{ab}$.

Now consider a unit normal to $\sum$ which we shall denote by n_α, i.e. $n_\alpha e_{(a)}{}^{\alpha} = 0$, $n_\alpha n^\alpha = 1$, and

$$
\frac{\delta n^\alpha}{\delta \theta^a} = \frac{\partial n^\alpha}{\partial \theta^o} + \Gamma^{\alpha}_{\mu\nu} n^\mu \frac{\partial x^\nu}{\partial \theta^a} \quad . \tag{5.1}
$$

Multiplying through (5.1) by n_α we obtain $n_\alpha \dfrac{\partial n_\alpha}{\delta \theta^a} = 0$ and thus we find that $\delta n^\alpha / \delta \theta^a$ is orthogonal to n_α and parallel to $\sum$. Thus we may now write down the definition of extrinsic curvature

$$
\frac{\delta n^\alpha}{\delta \theta^a} = - K^b{}_a e_{(b)}{}^{\alpha} \quad . \tag{5.2}
$$

Multiplying through (5.2) by ${e^{(c)}}_\alpha$ we obtain

$$K^c{}_a = -{e^{(c)}}_\alpha \frac{\delta n^\alpha}{\delta \theta^a} = n^\alpha \frac{\delta {e^{(c)}}_\alpha}{\delta \theta^a}$$

and therefore

$$K_{ab} = K_{ba} = -n_{\alpha|\beta}\, e_{(a)}{}^\alpha e_{(b)}{}^\beta \quad . \tag{5.3}$$

Now from the requirement that the metric be continuous up to arbitrary time translations across the boundary $\sum$ we have that f and curl $\underset{\sim}{\omega}$ must be continuous across $\sum$. Thus we may write the boundary conditions as

$$\begin{aligned} &[UU^*] = 0 \\ &\mathrm{Im}\ [U \frac{\partial U^*}{\partial n}] = 0 \quad . \end{aligned} \tag{5.4}$$

5.2 A Disk Solution

Assume that we have a disk on the x-y plane at z = 0. Now choose any U such that

$$\nabla^2 U = 0$$

and

$$U^*(\rho, z) = U(\rho, -z) \quad . \tag{5.5}$$

If we write U in the form

$$U = U_A\, e^{iU_p} \tag{5.6}$$

then

$$U^*(\rho,z) = U(\rho,-z) \longrightarrow \begin{cases} U_A(\rho,-z) = U_A(\rho,z) \\ \\ U_p(\rho,-z) = -U_p(\rho,z) \end{cases} \tag{5.7}$$

and thus UU* is continuous at $z = 0$. Also

$$\mathrm{Im}[U \frac{\partial U^*}{\partial n}] = 0$$

iff

$$[U_A{}^2 \frac{\partial U_p}{\partial z}] = 0$$

iff $\partial U_p/\partial z$ is continuous at $z = 0$.

We have thus shown that for any U such that (5.5), (5.6) and (5.7) hold we may construct a disk solution of the I.W.P. field.

If we let

$$U = 1 + \sum_{k=0}^{1} (A_{2k}q_{2k}(\tfrac{r}{R})p_{2k}(\cos\theta) + iA_{2k+1}q_{2k+1}(\tfrac{r}{R})p_{2k+1}(\cos\theta)) \tag{5.8}$$

where R is the radius of the disk, the coefficients A_k are real, (r,θ,ϕ) are oblate spheroidal coordinates and $q_k(ix)$ is a Legendre function of the second kind; $q_n(x) \equiv i^{n+1} Q_n(ix)$, then U is of the form discussed previously.

We may now proceed as follows to find the surface energy tensor, surface charge and current. Let n_α be the unit normal pointing upwards from the disk, then

$$n_\alpha = k\cos\theta\,\delta^1{}_\alpha = \varepsilon k\cos\rho\,\delta^1{}_\alpha \qquad (5.9)$$

where $k = (1 + R^2/q^2)^{-\frac{1}{2}}$ and $\varepsilon = +1$ and -1 on the upper and lower faces respectively. The extrinsic curvature of the disk is defined by (5.3). Due to the axial symmetry we have

$$\omega_a dx^a = \omega_\phi d\phi \qquad (5.10)$$

and thus once again we have[11]

$$\omega_\phi = \mathrm{Im}\int_\infty^{\rho^2} U\,\frac{\partial U^*}{\partial z}\,d(\rho^2) \qquad (5.11)$$

where by taking the definite integral we are fixing the additive gradient such that ω_ϕ goes to zero as ρ approaches infinity. On the plane of the disk we have $\theta = \pi/2$ for $R \leq \rho < \infty$ and $r = 0$ for $0 \leq \rho \leq R$. Thus the value of ω_ϕ at some point on the disk may be given in terms of the parameter θ which represents the distance from the center for $0 \leq \theta \leq \pi/2$. Therefore

$$\omega_\phi(0,\theta) = \mathrm{Im}\left\{\int_\infty^0 U\left(\frac{\partial U^*}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U^*}{\partial \theta}\frac{\partial \theta}{\partial z}\right)\Bigg|_{\theta=\pi/2} d(r^2)\right.$$

$$\left. + \int_{\pi/2}^{\theta} U\left(\frac{\partial U^*}{\partial r}\frac{\partial r}{\partial z} + \frac{\partial U^*}{\partial \theta}\frac{\partial \theta}{\partial z}\right)\Bigg|_{r=0} 2R^2\sin\theta\cos\theta\,d\theta\right\}$$

$$= \sum_{k=0}^{3} B_k\cos^{2k}\theta \qquad (5.12)$$

where

$$B_o = 2R\{((\frac{\pi^2}{16} + \frac{3}{4})A_1 - (\frac{\pi^2}{8} - \frac{5}{24})A_3)A_o$$

$$- (\frac{3}{16} A_1 + (\frac{105}{512}\pi^2 + \frac{3324}{768})A_3)A_2 - \frac{15}{32}\pi A_3\}$$

$$B_1 = -2R\{((\frac{\pi}{4} + (\frac{\pi^2}{8} - \frac{1}{2})A_o - (\frac{\pi^2}{32} - \frac{1}{4})A_2)A_1$$

$$- (\frac{9\pi}{8} + 3(\frac{3\pi^2}{16} - \frac{3}{4})A_o - \frac{3}{8}(\frac{3\pi^2}{16} - \frac{3}{2})A_2)A_3\}$$

$$B_2 = -2R\{(\frac{15}{32}\pi + \frac{5}{8}(\frac{3}{8}\pi^2 - \frac{3}{2})A_o - \frac{7}{8}(\frac{3}{16}\pi^2 - \frac{3}{2})A_2)A_3$$

$$+ \frac{3}{8}(\frac{\pi^2}{8} - 1)A_1A_2\}$$

$$B_3 = -\frac{5}{4}R(\frac{3}{16}\pi^2 - \frac{3}{2})A_2A_3$$

and

$$f^{-1}(0,\theta) = \sum_{k=0}^{3} C_k \cos^{2k}\theta \tag{5.13}$$

where

$$C_o = (1 + \frac{\pi}{2} A_o)^2 - \frac{\pi}{8}(1 + \frac{\pi}{2} A_o)A_2 + \frac{\pi^2}{64} A_2^2$$

$$C_1 = \frac{3}{8}\pi(1 + \frac{\pi}{2} A_o)A_2 - \frac{3}{32}\pi^2 A_2^2 + A_1^2 - \frac{9}{4} A_1A_3 + \frac{81}{16} A_3^2$$

$$C_2 = \frac{9}{64}\pi^2 A_2^2 + \frac{15}{4} A_1A_3 - \frac{135}{16} A_3^2$$

$$C_3 = \frac{225}{16} A_3^2 \quad .$$

Now for simplicity write

$$\left.\frac{\partial(f^{-1})}{\partial r}\right|_{r=0} \equiv \dot{f}^{-1} \quad ,$$

thus

$$\dot{f}^{-1} = \sum_{k=0}^{3} D_k \cos^{2k}\theta \tag{5.14}$$

$$D_o = \frac{2}{R}(1 + \frac{\pi}{2} A_o)A_o + \frac{1}{R}(1 + \frac{\pi}{4} A_o)A_2 - \frac{\pi}{2R} A_2^{\,2}$$

$$D_1 = - \frac{3}{R}(1 + \frac{\pi}{4} A_o - \frac{\pi}{2} A_2)A_2 + \frac{9\pi}{2R}(A_1 - \frac{9}{8} A_3)A_3 - \frac{\pi}{R} A_1^{\,2}$$

$$D_2 = - \frac{9}{32R} A_2^{\,2} - \frac{15\pi}{2R}(A_1 - \frac{9}{2} A_3)A_3$$

$$D_3 = \frac{225}{8} \frac{\pi}{R} A_3^{\,2}$$

and let a bar over all other quantities represent that quantity evaluated at $r = 0$.

The three metric of the disk may be written as

$$\bar{g}_{mn} dx^m dx^n = \bar{f}^{-1} \beta d\theta^2 + (\bar{f}^{-1}\eta - \bar{f}\bar{\omega}_\phi^2) d\phi^2 - 2\bar{f}\bar{\omega}_\phi d\phi dt - \bar{f}\, dt^2 \tag{5.15}$$

where $\beta = q^2 + R^2\cos^2\theta$, $\eta = (q^2 + R^2)\sin^2\theta$ and q = mass = charge of the disk. A straightforward calculation using (5.3) yields for the nonvanishing components of the extrinsic curvature tensor.

$$K_{22} = \varepsilon k\alpha \bar{f}\bar{\dot{f}}^{-1} \cos\rho$$

$$K_{33} = \varepsilon k\alpha\beta^{-1}\eta\bar{f}(\bar{\dot{f}}^{-1}(1 + \bar{f}^2\omega^2) - 2\bar{\omega}^\phi\bar{f}\bar{\dot{\omega}}_\phi)\cos\rho \tag{5.16}$$

$$K_{34} = \varepsilon k\alpha\beta^{-1}\bar{f}^2(\bar{f}\bar{\dot{f}}^{-1}\omega_\phi - \bar{\dot{\omega}}_\phi)\cos\rho$$

$$K_{44} = \varepsilon k\alpha\beta^{-1}\bar{f}^3\bar{\dot{f}}^{-1} \cos\rho$$

where $\alpha = q^2 + R^2$ and

$$\dot{\omega}_\phi = \frac{\partial}{\partial r}\,\omega_\phi\Big|_{r=0} = \sum_{k=0}^{3} E_k \cos^{2k}\theta \ , \tag{5.17}$$

$$E_o = 4(1 + \frac{\pi}{2} A_o - \frac{\pi}{8} A_2)(A_1 - A_3)$$

$$E_1 = 4(1 + \frac{\pi}{2} A_o)(A_1 - A_3) + \pi A_1 A_2 - \frac{3}{2}\pi A_2 A_3$$

$$E_2 = -\frac{\pi}{2} A_1 A_2 - \frac{29}{6} A_2 A_3$$

$$E_3 = -\frac{5}{2}\pi A_2 A_3 \quad .$$

If we consider all the gravitational and electromagnetic sources to be concentrated on the disk in the form of a surface layer then the surface energy tensor S_{ab} is given by[32]

$$-8\pi\, S_{ab} = [K_{ab}] - \bar{g}_{ab}\bar{g}^{cd}[K_{cd}] \tag{5.18}$$

in terms of the jump $[K_{ab}]$ in the extrinsic curvature on crossing the layer in the direction of n_α. Substituting equations (5.15) and (5.16) into (5.17) we obtain for the nonzero components of S_{ab}

$$S_{33} = \frac{k}{2\pi}\,\alpha\beta^{-1}\eta\, f^{-1}(\bar{f}\dot{\bar{f}}^{-1}\bar{\omega}^2 - \bar{\omega}^\phi\dot{\bar{\omega}}_\phi)\cos\rho \tag{5.19a}$$

$$S_{34} = \frac{k}{4\pi}\,\alpha\beta^{-1}\bar{f}^2(2\bar{f}\dot{\bar{f}}^{-1}\bar{\omega}_\phi - \dot{\bar{\omega}}_\phi)\cos\rho \tag{5.19b}$$

$$S_{44} = \frac{k}{2\pi}\,\alpha\beta^{-1}\bar{f}^3\dot{\bar{f}}^{-1}\cos\rho \tag{5.19c}$$

and from the relations[24]

$$S^a{}_b u^b = \sigma u^a$$

$$u_a u^a = -1 \tag{5.20}$$

(see Appendix B) we obtain

$$u^3 = \eta^{-1}\,\dot{\bar{\omega}}_\phi\{-1/2(\delta + (f^2\bar{\omega}^\phi - \eta^{-1})\dot{\bar{\omega}}_\phi{}^2)\}^{\frac{1}{2}}$$

$$u^4 = f^{-1}\delta\{-1/2(\delta + \dot{(}f^2\bar{\omega}^\phi - \eta^{-1})\dot{\bar{\omega}}_\phi{}^2\}^{\frac{1}{2}} \tag{5.21}$$

$$\sigma = -\frac{K}{4\pi}\,\alpha\beta^{-1}f^2(\dot{\bar{f}}^{-1} + (2f^2\omega^{\phi 2}(\dot{\bar{f}}^{-1})^2 + \dot{\bar{\omega}}_\phi{}^2) - \eta^{-1}\dot{\bar{\omega}}_\phi{}^2)^{\frac{1}{2}}) \tag{5.22}$$

where u^a is a unit timelike vector, σ is positive and

$$\delta = \dot{\bar{f}}^{-1} - \bar{f}\bar{\omega}^\phi\dot{\bar{\omega}}_\phi + ((\dot{\bar{f}}^{-1})^2 + 2f^2\bar{\omega}^{\phi 2}\dot{\bar{\omega}}_\phi{}^2 - \eta^{-1}\dot{\bar{\omega}}_\phi{}^2)^{\frac{1}{2}}. \tag{5.23}$$

The disk is thus composed of material having proper surface density σ and rotating with a velocity u^a. The geodesics on the surface of the disk are circles and thus there is no radial tension on the disk, which may be seen by the fact that the S_{a2} terms of the surface energy momentum tensor are zero.

Now consider arbitrary intrinsic coordinates ξ^2, ξ^3 and ξ^4 on the disk, we may extend these to a system of four-dimensional coordinates x^α by a Gaussian

construction[32,14], $x^a = \xi^a$, $x^1 = \pm$ (normal geodesic distance from the disk), so that $g_{1\alpha} = \delta_{1\alpha}$. Maxwell's equations

$$(4\pi)^{-1}\partial_\mu[(-g)^{\frac{1}{2}}F^{\lambda\mu}] = (-g)^{\frac{1}{2}}J^\lambda = (-g)^{\frac{1}{2}}j^\lambda\delta(x^1)$$

integrate to $[F^{\lambda 1}] = 4\pi\, j^\lambda$. Here j^λ is the surface current. Our result can be more conveniently written[14]

$$[e_{(a)}{}^\alpha F_{\alpha\beta}n^\beta] = 4\pi j_a \quad , \tag{5.24}$$

in which $j^\lambda = j^a e_{(a)}{}^\lambda$, and the left-hand side may now be evaluated in arbitrary four-dimensional coordinates, since it is a 4-scalar dependent only on the intrinsic coordinates ξ^a.

The angular momentum per unit mass of the disk may be calculated from the asymptotic form of the g_{34} term ($g_{34} \approx -2ma\sin^2\theta)/r$ $(r\to\infty)$,

$$a = \frac{1}{2m\sin^2\theta}\lim_{r\to\infty} rf\omega_\phi \quad . \tag{5.25}$$

To obtain the charge and current distribution on the disk we compute $F_{\alpha\beta}$ at $r = 0$ from eqn. (2.26). The nonvanishing components are

$$\begin{aligned} \bar{F}_{14} &= -\bar{F}_{23} = \operatorname{Im} \bar{f}^{-1}\bar{U}^{-2}(\partial_r U)\big|_{r=0} \\ \bar{F}_{13} &= \bar{F}_{24} = \operatorname{Im} \bar{f}^{-1}\bar{U}^{-2}(\partial_\theta U)\big|_{r=0} \,. \end{aligned} \tag{5.26}$$

Therefore we have from (5.9) and (5.19)

$$j_a = -\mathrm{Im}\,\frac{K}{2\pi}\, f^{-1}U^{-2}\alpha\beta^{-1}\cos\rho\,(0,\partial_\theta U,\partial_r U)\Big|_{r=0}$$

and(14)

$$j_a = qv_a$$

$$= -\frac{K}{4\pi}\, f^{-1}\alpha\beta^{-1}\,\rho\cos\rho\; v_a \qquad (5.28)$$

where v^a is a unit timelike vector

$$q = -\frac{K}{4\pi}\, f^{-1}\alpha\beta^{-1}\,\rho\cos\rho \qquad (5.29)$$

is the surface charge density and

$$\varepsilon\rho^2 = f\eta^{-1}\left(\partial_\theta\left(\frac{1}{U}-\frac{1}{U^*}\right)\right)^2 - 2f\omega^\phi\partial_\theta\left(\frac{1}{U}-\frac{1}{U^*}\right)\partial_r\left(\frac{1}{U}-\frac{1}{U^*}\right)$$

$$- (f^{-1} - f\omega^2)\left(\partial_r\left(\frac{1}{U}-\frac{1}{U^*}\right)\right)^2 \quad . \qquad (5.30)$$

Here ρ is always real since ε changes sign when f changes sign.

5.3 A Spherical Shell Approximation

In this section I work out the external and internal metric to second order in angular momentum for a spherical shell with an external metric calculated from the first order approximation in angular momentum of U for a Kerr-Newman source.

Thus we have

$$U_{ext} = 1 + \frac{q}{r} + i\,\frac{qa}{r^2}\,P_1(\cos\theta) \tag{5.31}$$

and the corresponding internal solution will have U_{int} of the form

$$U_{int} = A_o + A_1 r\,P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta). \tag{5.32}$$

We have no terms of the form $P_n(\cos\theta)$ for $n > 2$ since we neglect terms of $O(a^3)$ (order 3 of a), and the boundary conditions require the coefficients of these terms to be $O(a^3)$. Now note that as $\theta \to \pi-\theta$ we have $U_{ext} \to U^*_{ext}$ from which we conclude $U_{int} \to U^*_{int}$. But as $\theta \to \pi-\theta$ we have $P_1(\cos\theta) \to P_1(-\cos\theta) = -P_1(\cos\theta)$ and $P_2(\cos\theta) \to P_2(\cos\theta)$. Therefore we may conclude that A_o and A_2 are real and A_1 is imaginary. Now redefine A_1 such that

$$U_{int} = A_o + iA_1 r\,P_1(\cos\theta) + A_2 r^2 P_2(\cos\theta)\ . \tag{5.33}$$

We may now find A_o, A_1 and A_2 from the boundary conditions (5.3) which yield

$$(1 + \tfrac{q}{R})^2 + \frac{q^2a^2}{3R^4} + \frac{2}{3}\,\frac{q^2a^2}{R^4}\,P_2(\cos\theta)$$

$$= A_o^2 + \frac{A_1^2}{3}\,R^2 + (2A_2\,A_oR^2 + \tfrac{2}{3}\,A_1^2R^2)P_2(\cos\theta)$$

$$+ C_2^2R^2P_2^2(\cos\theta)$$

$$2(2 + \tfrac{q}{R})\frac{qa}{R^3}\,P_1(\cos\theta) = -2A_oA_1P_1(\cos\theta) - 2A_1A_2R^2P_1(\cos\theta)P_2(\cos\theta)$$

where R is the radius of the shell and equating coefficients of the Legendre polynomials we obtain:

$$A_2{}^2 = 0 \qquad (\rightarrow A_2{}^2 = 0(a^3)\ )$$

$$A_1A_2 = 0 \qquad (\rightarrow A_1A_2 = 0(a^3)\ )$$

$$A_o = \frac{1}{\sqrt{2}}\ [(1+\frac{q}{R})^2 + \frac{q^2a^2}{3R^4} + [((1+\frac{q}{R})^2 + \frac{q^2a^2}{3R^4})^2 + \frac{4}{3}(2+\frac{q}{R})^2\ \frac{q^2a^2}{R^4}]^{\frac{1}{2}}]^{\frac{1}{2}}$$

$$A_1 = -\frac{1}{A_o}\ (2+\frac{q}{R})\ \frac{qa}{R^3}$$

$$A_2 = \frac{1}{3A_o}\ (\frac{q^2a^2}{R^6} - A_1{}^2)\quad .$$

Now substituting (5.31) into (4.11) we obtain

$$\omega_{\phi_{ext}} = -qa\{\frac{qz^2}{(\rho^2+z^2)^2} + \frac{2z^2}{(\rho^2+z^2)^{3/2}} - \frac{q}{(\rho^2+z^2)} - \frac{2}{(\rho^2+z^2)^{\frac{1}{2}}}\} \tag{5.34}$$

and substituting (5.31), (4.10) and (5.34) into (2.25) we obtain the external metric.

If we now define, for the internal metric,

$$\omega_{\phi_{int}} = \mathrm{Im}\int_0^{\rho^2} U\ \frac{\partial U^*}{\partial z}\ d\rho^2 \tag{5.35}$$

we are fixing the additive gradient by having ω tend to zero as ρ approaches zero. Substituting (5.33) into (5.35) we obtain

$$\omega_{\phi\ int} = (A_1A_2z^2 - A_oA_1)\rho^2 + \frac{1}{4}\ A_1A_2\rho^4 \tag{5.36}$$

and substituting (5.33), (4.10) and (5.36) into (2.25) we obtain the internal metric.

5.4 An Exact Internal Solution

In this section I shall set up the boundary conditions for the exact internal metric for a spherical shell with an external metric calculated from the first order approximation in angular momentum of U for a Kerr-Newman source.

Thus we have

$$U_{ext} = 1 + \frac{q}{r} + i\,\frac{qa}{r^2}\,P_1(\cos\theta) \tag{5.37}$$

and once again we conclude $U_{int} \to U^*_{int}$ as $\theta \to \pi-\theta$ as in section 5.3. Thus we have the corresponding internal solution U_{int} in the form

$$U_{int} = \sum_{n=0}^{\infty} (A_{2n}r^{2n}P_{2n}(\cos\theta) + iA_{2n+1}r^{2n+1}P_{2n+1}(\cos\theta)). \tag{5.38}$$

One may easily verify that the solution must be an infinite series by terminating it at some r, we then find that $a_m=0 \;\forall m \ni 2<m\leq n$ and that there does not exist a solution of the form

$$U_{int} = A_o + iA_1\,r\,P_1(\cos\theta)\ . \tag{5.39}$$

We now wish to calculate the A_n's from the boundary conditions (5.3). Substituting (5.37) and (5.39) into (5.4) we obtain

$$(1+\frac{q}{r})^2 + \frac{q^2a^2}{3r^4} + \frac{2}{3}\frac{q^2a^2}{r^4} P_2(\cos\theta) = 2\sum_{n=0}^{\infty} A_o A_{2n} R^{2n} P_{2n}(\cos\theta)$$

$$+ \frac{1}{2}\sum_{s=1}^{\infty}\sum_{n=1}^{\infty}(-1)^s((-1)^n+(-1)^s)A_n A_s R^{n+s} P_n(\cos\theta)P_s(\cos\theta)$$

$$(5.40)$$

and

$$2(2+q)\frac{qa}{R^3}P_1(\cos\theta) = 2\sum_{s=0}^{\infty}\sum_{n=0}^{\infty}(2(s-n)-1)A_{2s}A_{2n+1} \times$$

$$\times R^{2(n+s)} P_{2s}(\cos\theta)P_{2n+1}(\cos\theta) \ .$$

Now for simplicity I will introduce Clebsch-Gordan coefficients in order to rewrite the terms with products of Legendre Polynomials in the boundary conditions as series of Legendre Polynomials. Thus we have[31]

$$P_m(\cos\theta)P_n(\cos\theta) = \sum_{k=0}^{m+n} C_{m,n;k}\, P_k(\cos\theta) \qquad (5.41)$$

$$C_{m,n;k} = \begin{cases} (-1)^{g+k}(2k+1)^{\frac{1}{2}}\Delta(mnk)g![(g-m)!(g-n)!(g-k)!] & 2g \text{ even} \\ 0 & 2g \text{ odd} \end{cases}$$

where $2g = m+n+k$, and

$$\Delta(mnk) = \frac{(m+n-k)!(m+k-n)!(n+k-m)!}{(m+n+k+1)!} \ .$$

The boundary conditions (5.41) may now be written as

$$(1+\frac{q}{R})^2 + \frac{q^2a^2}{3R^4} + \frac{2}{3}\frac{q^2a^2}{R^4} P_2(\cos\theta)$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{k=0}^{m+n} A_{2m}A_{2n}R^{2(m+n)}C_{2m,2n;2k}P_{2k}(\cos\theta)$$

$$+ \sum_{k=0}^{m+n+1} R^{2(m+n+1)}C_{2m+1,2n+1;2k}P_{2k}(\cos\theta)$$

$$2(2+\frac{q}{R})\frac{qa}{R^3}P_1(\cos\theta) = 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m+n}(2(m-n)-1)\times$$

$$\times A_{2m}A_{2n+1}R^{2(m+n)}C_{2n+1,2m;2k+1}P_{2k+1}(\cos\theta)$$

and equating the coefficients of the Legendre Polynomials we obtain

$$\sum_{m=0}^{\infty} A_m^2R^{2m}C_{m,m;o} = (1+\frac{q}{R})^2 + \frac{q^2a^2}{3R^4}$$

$$\sum_{m=0}^{\infty} A_mA_{m+1}R^{2m}C_{m+1,m;1} = -(2+\frac{q}{R})\frac{qa}{R^3}$$

$$\sum_{m=0}^{\infty}(A_{m+1}^2R^{2(m+1)}C_{m+1,m+1;2} + 2A_mA_{m+2}R^{2(m+1)}C_{m,m+2;2}) = \frac{2}{3}\frac{q^2a^2}{R^4}$$

$$\sum_{m=0}^{\infty}(A_{m+1}A_{m+2}R^{2(m+1)}C_{m+1,m+2;3} + A_mA_{m+3}R^{2(m+1)}C_{m,m+3;3}) = 0$$

$$\vdots$$

$$\sum_{m=0}^{\infty}(A_{m+k}^2R^{2(m+k)}C_{m+k,m+k;2k} + 2A_{m+k-1}R^{2(m+k)}C_{m+k-1,m+k+1;2k}$$

$$\ldots + 2A_mA_{m+2k}R^{2(m+k)}C_{m,m+2k;2k}) = 0$$

$$\sum_{m=0}^{\infty}(A_{m+k}A_{m+k+1}R^{2(m+k)}C_{m+k,m+k+1;2k+1} + A_{m+k-1}A_{m+k+2}R^{2(m+k)}\times$$

$$\times C_{m+k-1,m+k+2;2k+1} + \ldots + A_mA_{m+2k+1}R^{2(m+k)}C_{m,m+2k+1;2k+1}) = 0$$

$$\vdots$$

This system of equations specifies the coefficients. A_n exactly however appears to be unsolvable.

CHAPTER 6

THE INTERNAL KERR-NEWMAN SOLUTION

6.1 The Thin Shell Solution

I wish to now explain a method by which the internal Kerr-Newman metric with charge equal to mass of an infinitely thin shell may be worked out.

We have

$$U_{ext} = 1 + \frac{q}{\sqrt{r^2 - 2iar\cos\theta - a^2}} \tag{6.1}$$

and thus we may see that as $\theta \to \pi-\theta$ we have $U_{ext} \to U^*_{ext}$ and therefore we assume $U_{int} \to U^*_{int}$. Also assume U_{int} may be written in the form

$$U_{int} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \ . \tag{6.2}$$

Therefore as in Chapter 5 section 4 we may redefine A_n and write

$$U_{int} = \sum_{n=0}^{\infty} [A_{2n} r^{2n} P_{2n}(\cos\theta) + iA_{2n+1} r^{2n+1} P_{2n+1}(\cos\theta)] \tag{6.3}$$

where the A_n's are all real.

Applying the boundary conditions (5.3) we have

$$1 + 2q\left(1 + \frac{q}{R}\right) \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{R^{2n+1}} P_{2n}(\cos\theta) + \frac{q^2}{2} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{n=0}^{s+t} \times$$

$$\times\ (-1)^{s+t/2} \frac{((-1)^s + (-1)^t)}{R^{s+t+2}} a^{s+t} C_{s,t;2n} P_{2n}(\cos\theta)$$

$$= \frac{1}{2} \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{n=0}^{s+t} (-1)^s ((-1)^s + (-1)^t) A_s A_t R^{s+t} C_{s,t;2n} P_{2n}(\cos\theta)$$

$$2q \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n+1} 2(n+1)}{R^{2n+3}} P_{2n+1}(\cos\theta)$$

$$+ 2q^2 \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{n=0}^{s+t} \frac{(-1)^{s+t}(2(s-t)+1)a^{2(s+t)+1}}{R^{2(s+t+2)}} \times$$

$$\times C_{2s+1,2t;2n+1} P_{2n+1}(\cos\theta)$$

$$= 2 \sum_{s=0}^{\infty} \sum_{t=0}^{\infty} \sum_{n=0}^{s+t} (2(s-t)-1) A_{2t} A_{2s+1} R^{2(s+t)} \times$$

$$\times C_{2s+1,2t;2n+1} P_{2n+1}(\cos\theta) \;.$$

Now we wish to solve for the coefficients A_n. This may be done by taking successively higher order approximations in angular momentum per unit mass and solving for the A_n in each case until we are able to find their values for an arbitrary order of angular momentum per unit mass, k say. We then take the limit as k goes to infinity and obtain the exact form of the coefficients A_n. This procedure may be carried out as follows: First equating coefficients of $P_n(\cos\theta)$ and using properties of Clebsch-Gordan coefficients we obtain

$$\sum_{n=0}^{\infty} A_n^2 R^{2n} C_{m,n;o} = 1 + \frac{2q}{R}(1+\frac{q}{R}) + \frac{q^2}{R^2} \sum_{n=0}^{\infty} (-1)^n (\frac{q}{R})^n C_{n,n;o}$$

$$\sum_{n=0}^{\infty} A_n A_{n+1} R^{2n} C_{n+1,n;1} = -2q \frac{q}{R^3} - \frac{q^2}{R^3} \sum_{n=0}^{\infty} (\frac{q}{R})^{4n+1} C_{2n+1,2n;1}$$

$$\vdots$$

$$\sum_{n=0}^{\infty}(A_{n+k}^2 R^{2(n+k)} C_{n+k,n+k;2k} + 2A_{n+k-1}A_{n+k+1}R^{2(n+k)} \times$$

$$\times\ C_{n+k-1,n+k+1;2k} \ldots + 2A_n A_{n+2k} R^{2(n+k)} C_{n,n+2k;2k})$$

$$= (-1)^k\, 2q\,(1 + \frac{q}{R})\ \frac{q^{2k}}{R^{2k+1}} + \frac{q^2}{R^2}\sum_{n=0}^{\infty}(-1)^{n+k}(\frac{q}{R})^{2(n+k)} \times$$

$$\times\ (C_{n+k,n+k;2k} + 2C_{n+k-1,n+k+1;2k} + \ldots + 2C_{n,n+2k;2k})$$

$$\sum_{n=0}(A_{n+k}A_{n+k+1}R^{2(n+k)}C_{n+k,n+k+1;2k+1} + A_{n+k-1}A_{n+k+2}R^{2(n+k)} \times$$

$$\times\ C_{n+k-1,n+k+2;2k+1} + \ldots + A_n A_{n+2k+1} R^{2(n+k)} C_{n,n+2k+1;2k+1}))$$

$$= (-1)^k\ 2q\frac{q^{2k+1}(k+1)}{R^{2k+3}} + q^2\sum_{n=0}^{\infty}\frac{q^{4(n+k)+1}}{R^{4(n+k)+2}}((1-4k) \times$$

$$\times\ C_{n,n+2k+1;2k+1} + (1-4k+4)C_{2n+3,2n+2(k-1);2k+1} + \ldots +$$

$$(1+4k)C_{2n+2k+1,2n;2k+1})$$

$\vdots$

Now write A_n to the k^{th} order in angular momentum per unit mass as $A_n^{(k)}$. Assuming the angular momentum per unit mass to be small and thus neglecting all non static terms we find

$$A_o^{(o)} = [\frac{1}{C_{o,o;o}}(1 + 2\ \frac{q}{R}(1 + \frac{q}{R})) + \frac{q^2}{R^2}\ C_{o,o;o}]^{\frac{1}{2}}$$

$$A_k^{(o)} = 0 \qquad\qquad \forall\ k > 0$$

Similarly

$$A_o^{(1)} = [\frac{1}{C_{o,o;o}}(1 + 2\,\frac{q}{R}(1 + \frac{q}{R})) + \frac{q^2}{R^2}\,C_{o,o;o}]$$

$$A_1^{(1)} = -\,\frac{(qa/R^3)}{A_o^{(1)}C_{1,o;1}}\,[2 - \frac{q}{R}\,C_{1,o;1}]$$

$$A_k^{(1)} = 0 \qquad \forall\, k > 1$$

and

$$A_o^{(2)} = \Bigg\{1 + \frac{2q}{R}(1 + \frac{q}{R}) + \frac{q^2}{R^2}(C_{o,o;o} + \frac{q^2}{R^2}\,C_{1,1;o})$$

$$+ [(1 + \frac{2q}{R}(1 + \frac{q}{R}) + \frac{q^2}{R^2}(C_{o,o;o} + \frac{q^2}{R^2}\,C_{1,1;o}))^2$$

$$+ 4C_{o,o;o}C_{1,1;o}(\frac{qa}{R^2C_{1,o;1}}[2 - \frac{q}{R}\,C_{1,o;1}])^2]^{\frac{1}{2}}\,\frac{1}{2C_{o,o;o}}\Bigg\}^{\frac{1}{2}}$$

$$A_1^{(2)} = -\,\frac{(qa/R^3)}{A_o^{(2)}C_{1,o;1}}\,[2 - \frac{q}{R}\,C_{1,o;1}]$$

$$A_2^{(2)} = -\,\frac{1}{2A_o^{(2)}C_{o,2;2}}[A_1^{(2)\,2}R^2C_{1,1;2} + 2q(1 + \frac{q}{R})\frac{q^2}{R^3} + \frac{q^2}{R^2}\,C_{1,2;2}]$$

$$A_k^{(2)} = 0 \qquad \forall\, k > 2$$

and

$$A_o^{(3)} = A_o^{(2)}$$

$A_1^{(3)}$ = (a complicated expression coming from the solution of a general cubic)

$$A_2^{(2)} = - \frac{1}{2A_o^{(3)} R^4 C_{o,2;2}} [A_1^{(2)\,2} R^2 C_{1,1;2} + 2q(1+\frac{q}{R})\frac{q^2}{R^3} + \frac{q^2}{R^2} C_{1,2;2}]$$

$$A_3^{(3)} = - \frac{1}{A_o^{(3)} R^2 C_{o,3;3}} [A_1 A_2 R^2 C_{1,2;3} + 4q \frac{q^3}{R^5}]$$

$$A_k^{(3)} = 0 \qquad \forall\, k > 3 .$$

After this point the equations for the coefficients become too complicated to solve explicitly. Thus from these approximations one must guess what the general term $A_o^{(k)}$ looks like and how $A_\ell^{(k)}$ may be written as a function of $A_o^{(k)}$ and then check these guesses in the higher order equations.

Once the general coefficients are found then we have ω_ϕ given by

$$\omega_\phi = - \sum_{n=0}^{\infty} \sum_{m=n+1}^{\infty} A_n A_m P^{2n}$$

and (6.4)

$$f = (UU^*)^{-1} .$$

Substituting (6.3) and (6.4) into (2.25) and (2.26) we obtain the interior Kerr-Newman solution for charge = mass.

6.2 An Interior Extension of the I.W.P. Fields[13]

For a more complete understanding one would like to examine the interior (i.e. non-vacuum) extension of the I.W.P. field. Such an extension is, of course, not unique. Here is presented one possible extension which is mathematically simple but has the disadvantage of requiring nonvanishing magnetic charge density.

Let $\rho(x,y,z)$ be an arbitrary real function, and $U(x,y,z)$ a complex solution of

$$\nabla^2 U = -4\pi\rho U \; . \tag{6.5}$$

Since U satisfies (2.4a), the I.W.P. prescription (eqs. 2.24 - 2.26) for generating a metric and an electromagnetic field can be carried through exactly as before. We now obtain, for the electric and magnetic currents,

$$J^{\alpha}_{(el)} + iJ^{\alpha}_{(mag)} = \frac{1}{4\pi}\,\mathcal{F}^{\alpha\beta}{}_{|\beta} = \rho U^{-1} f^{\frac{1}{2}} v^{\alpha} \tag{6.6}$$

and the Einstein field equations[5] yield

$$T^{\alpha\beta} = f\rho v^{\alpha} v^{\beta} \tag{6.7}$$

for the material energy tensor (i.e. the contribution of the electromagnetic field has been subtracted out).

The interior extension therefore represents a static distribution of dually charged dust. The source of angular momentum is the circulating poynting vector

arising from the crossed electric and magnetic fields. The static counterpart has been previously given by Das[4]. The question whether other, more "physical" sources, free of magnetic charge, can be constructed for the I.W.P. fields (at least in certain instances) has been settled in the affirmative for a single Kerr-Newman source[34].

6.3 A Thick Shell Solution

In this section I will show how one may match a given external solution to a given internal solution across a thick spherical shell by specifying the metric within the shell. In specific I develop the method by which one may match the external Kerr-Newman solution, where

$$U_{ext} = 1 + \frac{q}{\sqrt{\rho^2 + (z-ia)^2}} , \tag{6.8}$$

to the interior solution calculated from

$$U_{int} = 1 + \frac{(q \frac{R_1}{ia})}{\sqrt{\rho^2 + (z + i \frac{R_1^2}{a})}} . \tag{6.9}$$

Here R_1 is the inner radius of the shell and R_2 is the outer radius.

First we must find U_T (the function U that generates the metric within the shell) such that it is a

solution of (6.5) and satisfies the boundary conditions (5.4) at $r = R_1$ and $r = R_2$.

For U_T $(R_1 < r < R_2)$ assume

$$U_T = \sum_{n=0}^{\infty} f_n(r) P_n(\cos\theta) \quad . \tag{6.10}$$

Therefore

$$\nabla^2 U = \sum_{n=0}^{\infty} \left(f_n'' + \frac{2}{r} f_n' - \frac{n(n+1)}{r^2} f_n\right) P_n(\cos\theta) \tag{6.11}$$

where the primes denote differentiation with respect to r. Now from (6.5) and (6.11) we obtain

$$f_n''(r) + \frac{2}{r} f_n'(r) + \left[4\pi\rho - \frac{n(n+1)}{r^2}\right] f_n(r) = 0 \quad . \tag{6.12}$$

Let $4\pi\rho = \ell(\ell+1)/r^2$, ℓ = constant, then we obtain as the general solution of equation (6.11)

$$f_n = A_n \, r^{P_{1n}} + B_n \, r^{P_{2n}}$$

where

$$P_{1n} = \frac{-1 + \sqrt{1-4(\ell-n)(\ell+n+1)}}{2} \;, \quad P_{2n} = \frac{-1 - \sqrt{1-4(\ell-n)(\ell+n+1)}}{2} \;.$$

Therefore

$$U_T = \sum_{n=0}^{\infty} (A_n \, r^{P_{1n}} + B_n \, r^{P_{2n}}) P_n(\cos\theta) \quad . \tag{6.13}$$

Solving for the complex constants A_n and B_n becomes rather complicated, since we now have boundary

conditions on two surfaces instead of one surface as in section 6.1. The solution however carries through in the same manner outlined in section 6.1, however upon writing out the boundary conditions one may verify that a solution exists without actually solving for it.

Substituting (6.9) into (4.11) we obtain

$$\omega_{\phi_{int}} = +2\ \mathrm{Im}\left(\frac{\alpha \overset{*}{z}_1}{\overset{*}{R}} + \frac{\alpha \overset{*}{z}_1}{z^2 - \overset{*}{z}{}_1^2}\left[\frac{R}{\overset{*}{R}} - 1\right]\right) \tag{6.14}$$

where

$$\alpha = \frac{qR_1}{ia} \quad ,$$

$$z_1 = z + i\,\frac{R_1^2}{a} \quad ,$$

and

$$R^2 = \rho^2 + \left(z + i\,\frac{R_1^2}{a}\right)^2 \quad .$$

Thus substituting (6.9), (4.10) and (6.13) into (2.25) and (2.26) we obtain the solution of the Einstein-Maxwell equations for the complex function (6.9). Furthermore we have shown that this interior solution may be matched to the exterior Kerr-Newman solution across a thick shell.

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33. ω_ϕ is the only non-vanishing component due to the axial symmetry.

34. There is a difference in sign due to the choice of sign in the original form of the metrics.

APPENDIX A

THE CALCULATION OF ω_ϕ IN SECTION 4.3 AND A CHECK OF ITS CONSISTENCY WITH THE GENERAL CONDITION FOR THE ABSENCE OF STRUTS AND MAGNETIC POLES

From (4.11) and (4.21) we have

$$\omega_\phi = \mathrm{Im} \int U \frac{\partial U^*}{\partial z} \, d(\rho^2) \tag{A.1}$$

and

$$U = 1 + \sum_{A=1}^{N} \frac{q_A}{R_A} \tag{A.2}$$

where $R_A = (\rho^2 + z_A^2)^{\frac{1}{2}}$ and $z_A = z + b_A + ia_A$. Thus

$$\omega_\phi = - \mathrm{Im}\int (1 + \sum_{B=1}^{N} \frac{q_B}{R_B}) \, (\sum_{A=1}^{N} \frac{q_A \overset{*}{z}_A}{\overset{*}{R}{}_A^3}) \, d(\rho^2)$$

$$= - \mathrm{Im} \int \sum_{A=1}^{N} \{\frac{q_A \overset{*}{z}_A}{\overset{*}{R}{}_A^3} + \sum_{B=1}^{N} \frac{q_A q_B \overset{*}{z}_A}{\overset{*}{R}{}_A^3 R_B}\} \, d(\rho^2)$$

$$= 2 \ \mathrm{Im} \sum_{A=1}^{N} \{\frac{q_A \overset{*}{z}_A}{\overset{*}{R}_A} + \sum_{B=1}^{N} \frac{q_A q_B \overset{*}{z}_A}{\overset{*}{z}{}_A^2 - z_B^2} \frac{R_B}{\overset{*}{R}_A}\} + C$$

$$= 2 \ \mathrm{Im} \sum_{A=1}^{N} \frac{q_A \overset{*}{z}_A}{\overset{*}{R}_A} + \sum_{A=1}^{N} \sum_{B=1}^{N} q_A q_B \frac{\rho^2 + z_A \overset{*}{z}_B}{(z_A - \overset{*}{z}_B) R_A \overset{*}{R}_B} + C \tag{A.3}$$

and $\dfrac{\partial \omega}{\partial z} = 2\rho \ \mathrm{Im} \ U^* \dfrac{\partial U}{\partial \rho} \rightarrow C$ is independent of z.

Let

$$C = -\sum_{A=1}^{N} \frac{q_A^2}{2a}$$

then

$$\omega_\phi(0,z) = \sum_{A=1}^{N} \sum_{\substack{B=1 \\ B\neq A}}^{N} \frac{q_A q_B (a_A + a_B)}{(b_A - b_B)^2 + (a_A + a_B)^2} \frac{1}{\varepsilon_A \varepsilon_B} \tag{A.4}$$

where

$$\varepsilon_A = \begin{cases} +1 & \text{for} \quad z - b_A > 0 \\ -1 & \text{for} \quad z - b_A < 0 \end{cases} \quad .$$

Now we wish to verify that

$$\sum_{A=1}^{N} \sum_{B=1}^{N} q_A q_B A_{AB} \frac{1}{\varepsilon_A \varepsilon_B} = 0 \tag{A.5}$$

is equivalent to the general condition for the absence of struts and magnetic poles where

$$A_{AB} = \frac{a_A + a_B}{(b_A - b_B)^2 + (a_A + a_B)^2} \quad .$$

We may write (A.5) as the N equations

$$\sum_{A=1}^{N} \sum_{\substack{B=1 \\ B\neq A}}^{N} q_A q_B A_{AB} = 0 \tag{A.6i}$$

$$\vdots$$

$$\sum_{A=1}^{k-1} \sum_{\substack{B=1 \\ B\neq A}}^{k-1} q_A q_B A_{AB} - 2 \sum_{A=1}^{k-1} \sum_{B=k}^{N} q_A q_B A_{AB} + \sum_{A=k}^{N} \sum_{\substack{B=1 \\ B\neq A}}^{N} q_A q_B A_{AB} = 0 \tag{A.6k}$$

$$\vdots$$

$$\sum_{A=1}^{N-1} \sum_{B=1}^{N-1} q_A q_B A_{AB} - 2 \sum_{A=1}^{N-1} q_A q_N A_{AN} = 0 \tag{A.6N}$$

then (A.6 k-1) minus (A.6 k) yields

$$\sum_{A=1}^{N} q_A A_{Ak-1}\,\epsilon\,(A-(k-1)) = 0 \;. \tag{A.7}$$

If we now consider (A.7) $\forall\, k \in \{2,\ldots,N\}$ and (A.6N) minus (A.6 i) we obtain

$$\sum_{A=1}^{N} q_A A_{AB}\,\epsilon\,(A-B) = 0 \qquad \forall\, B \in \{1,\ldots,N\} \tag{A.8}$$

which is the general condition for the absence of struts and magnetic poles.

APPENDIX B

THE MASS DENSITY AND VELOCITY OF ROTATION OF THE DISK SOLUTION IN SECTION 5.2

From the relations (5.20) we have

$$\left.\begin{aligned} S^3_3 u^3 + S^3_4 u^4 &= \sigma u^3 \\ S^4_3 u^3 + S^4_4 u^4 &= \sigma u^4 \end{aligned}\right\} \qquad \text{(B1)}$$

$$u_3 u^3 + u_4 u^4 = \varepsilon \quad . \qquad \text{(B.2)}$$

Equations (B.1) may be rewritten as

$$S^3_4 u^4 u^4 + (S^3_3 - S^4_4) u^3 u^4 - S^4_3 u^3 u^3 = 0$$

and thus

$$u^4 = \frac{S^4_4 - S^3_3 \pm \sqrt{(S^4_4 - S^3_3)^2 + 4 S^3_4 S^4_3}}{2 S^3_4}\, u^3 \quad . \qquad \text{(B.3)}$$

Equation (B.2) may be rewritten as

$$g_{44} u^4 u^4 + 2 g_{34} u^3 u^4 + g_{33} u^3 u^3 - \varepsilon = 0$$

and thus

$$u^4 = - \frac{g_{34} u^3 \pm \sqrt{g^2_{34} u^3 u^3 - g_{44}(g_{33} u^3 u^3 - \varepsilon)}}{g_{44}} \quad . \qquad \text{(B.4)}$$

Now equating (B.3) and (B.4) we obtain for the velocity in the ϕ direction

$$u^3 = \left\{ \varepsilon \,/ [\{ g_{44} \left(\frac{S^4_4 - S^3_3 + \quad (S^4_4 - S^3_3)^2 + 4S^3_4 S^4_3}{2S^3_4} \right) + g_{34} \} \right.$$

$$\left. - g^2_{34} - g_{44} g_{33}] \right\}^{\frac{1}{2}} \tag{B.5}$$

where we have taken the "+" root of $((S^4_4 - S^3_3)^2 + 4S^3_4 S^4_3)$ since u^3 must vanish in the static solution. u^4 may now be calculated by substituting (B.5) into (B.3) again taking the "+" sign in front of the square root in (B.3). Now the mass density σ is given by the equation

$$\sigma = S^3_3 + S^3_4 u^4 / u^3 \quad .$$

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